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Characterization of initial data in the homgeneous Besov space for solutions in the Serrin class of the Navier-Stokes equations

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1 Introduction and Results.

Let us consider the Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$;

$$\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla \pi = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = a \quad \text{in } \mathbb{R}^n, \end{cases}$$

where $u = u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $\pi = \pi(x,t)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and the time $t \in (0,T)$, respectively, while $a = a(x) = (a_1(x), \cdots, a_n(x))$ is the given initial velocity vector. It is well-known that (N-S) is invariant under such a change of scaling as $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ and $\pi_\lambda(x,t) = \lambda^2 \pi(\lambda x, \lambda^2 t)$ for all $\lambda > 0$. The Banach space $\mathcal{Y}$ of functions with the space and time variables with the norm $\| \cdot \|_{\mathcal{Y}}$ is called *scaling invariant* to (N-S) if it holds that $\|u_\lambda\|_{\mathcal{Y}} = \|u\|_{\mathcal{Y}}$ for all $\lambda > 0$. Since the corresponding scaling law to the initial data a is $a_\lambda(x) = \lambda a(\lambda x)$, it is suitable to solve (N-S) in the Banach space X for a with such a property as $\|a_\lambda\|_X = \|a\|_X$ for all $\lambda > 0$. Since the pioneer work of Fujita-Kato [8], there have been a number of results to

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enlarge the space X such as $H^{\frac{n}{2}-1}(\mathbb{R}^n)$, $L^n(\mathbb{R}^n)$, $\dot{B}^{-1+n/p}_{p,\infty}(\mathbb{R}^n)$ with $n < p < \infty$ and BMO^{-1} which are monotonically increasing. See e.g., Kato [11], Giga-Miyakawa [10], Kozono-Yamazaki [16], Cannone [4], Cannone-Planchon [5] and Koch-Tataru [12]. Amann [1] has established a systematic treatment of strong solutions in various function spaces such as Lebesgue space $L^p(\Omega)$, Bessel potential space $H^{s,p}(\Omega)$, Besov sapce $B^s_{p,q}(\Omega)$ and Nikol'skii space $N^{s,p}(\Omega)$ in general domains Ω. On the other hand, in the critical case $p = \infty$, ill-posedness in the sense of break-down of continuous dependence of the solution mapping for $a \in \dot{B}^{-1}_{\infty,q}$ of (N-S) was proved by Bourgain-Pavlović [3] for $q = \infty$, Yoneda [19] for $2 < q < \infty$ and Wang [18] for $1 \leqq q \leqq 2$.

The purpose of this article is to characterize the optimal space of the initial data a for existence of mild solution u of (N-S) in the generalized Serrin class $L^{\alpha,q}(0,\infty; L^r(\mathbb{R}^n))$ for $\frac{2}{\alpha} + \frac{n}{r} = 1$ with $n < r \leqq \infty$. In a bounded domain Ω, a similar investigation has been observed by Farwig-Sohr-Varnhorn [7] and Farwig-Sohr [6]. Indeed, they proved that for $a \in L^2_\sigma(\Omega)$, with $L^r_\sigma(\Omega)$ for $1 < r < \infty$ denoting the closure of compactly supported solenoidal vector fields in $L^r(\Omega)$, the mild solution u of (N-S) with the homogeneous boundary condition belongs to $L^\alpha(0,T; L^r(\Omega))$ with some $0 < T \leqq \infty$ for such α and r as above *if and only if* it holds that

$$\int_0^\delta \|e^{-tA_2}a\|^\alpha_{L^r(\Omega)} dt < \infty \tag{1.1}$$

for some $0 < \delta \leqq T$, where A_2 denotes the Stokes operator in $L^2_\sigma(\Omega)$. By the real interpolation, such an initial data $a \in L^2_\sigma(\Omega)$ satisfying the condition (1.1) is characterized as $a \in B^{-\frac{2}{\alpha}}_{r,\alpha}(\Omega) = B^{\frac{2}{\alpha}}_{r',\alpha'}(\Omega)^*$, where $B^{\frac{2}{\alpha}}_{r',\alpha'}(\Omega) = (L^{r'}_\sigma(\Omega), D(A_{r'}))_{\frac{1}{\alpha},\alpha'}$ for $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ with $D(A_{r'})$ denoting the domain of the Stokes operator $A_{r'}$ in $L^{r'}_\sigma(\Omega)$. Since the Stokes semigroup $\{e^{-tA_2}\}_{t\geq 0}$ in a bounded domain Ω exhibits an exponential decay in $L^r(\Omega)$ as $t \to \infty$, we see easily that the condition (1.1) is equivalent to

$$\int_0^\infty \|e^{-tA_2}a\|^\alpha_{L^r(\Omega)} dt < \infty. \tag{1.2}$$

On the other hand, in the whole space $\mathbb{R}^n$, we *cannot* expect any exponential decay of $\{e^{t\Delta}\}_{t\geq 0}$ in $L^r(\mathbb{R}^n)$. To get around such difficulty, we shall establish a sharp estimate

$$\Big\| \|e^{t\Delta}a\|_{\dot{B}^0_{r,1}(\mathbb{R}^n)} \Big\|_{L^{\alpha,q}(0,\infty)} \leqq C\|a\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{R}^n)} \tag{1.3}$$

for all $a \in \dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ with $n < p < \infty$ and $1 \leqq q \leqq \infty$ provided $\frac{2}{\alpha} + \frac{n}{r} = 1$ with $p \leqq r \leqq \infty$. Here $L^{\alpha,q}(0,\infty)$ denotes the Lorentz space on $(0,\infty)$. Since we are also successful to derive the continuous bilinear estimate of the Duhamel term $\displaystyle\int_0^t P\nabla \cdot e^{(t-\tau)\Delta} u \otimes v(\tau)d\tau$ for $u, v \in L^{\alpha,q}(0,\infty; \dot{B}^0_{r,1}(\mathbb{R}^n))$, it follows from (1.3) that there exists a unique global mild solution $u \in L^{\alpha,q}(0,\infty; \dot{B}^0_{r,1}(\mathbb{R}^n))$ provided a is sufficiently small in $\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{R}^n)$. It should be emphasized that such a global existence result for small initial data can be obtained because such an estimate as (1.3) holds on the whole interval $(0,\infty)$. Notice that our class is stronger than the Serrin class because it holds a continuous embedding $\dot{B}^0_{r,1}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$. Now, a natural question arises whether the estimate (1.3) is optimal or not. It will be clarified that if $a \in \mathcal{S}'$ satisfies

$e^{t\Delta}a \in L^{\alpha,q}(0,\infty; L^r(\mathbb{R}^n))$ for $\frac{2}{\alpha} + \frac{n}{r} = 1$ with $n < r \leqq \infty$ and for $1 < q \leqq \infty$ ($\mathcal{S}'$ denotes the class of temperated distribution), then it holds that $a \in \dot{B}^{-1+\frac{n}{r}}_{r,q}(\mathbb{R}^n)$ with the estimate

$$\|a\|_{\dot{B}^{-1+\frac{n}{r}}_{r,q}(\mathbb{R}^n)} \leqq C \left\| \|e^{t\Delta}a\|_{L^r(\mathbb{R}^n)} \right\|_{L^{\alpha,q}(0,\infty)} \tag{1.4}$$

Since the continuous bilinear estimate of the Duhamel term holds for $u,\ v \in L^{\alpha,q}(0,\infty; L^r(\mathbb{R}^n))$, we conclude from (1.4) that the mild solution u of (N-S) belongs to the Serrin class $L^{\alpha,q}(0,\infty; L^r(\mathbb{R}^n))$ for $\frac{2}{\alpha} + \frac{n}{r} = 1$ with $n < r \leqq \infty$ and for $1 < q \leqq \infty$, then the initial data a necessarily satisfies that $a \in \dot{B}^{-1+\frac{n}{r}}_{r,q}(\mathbb{R}^n)$.

To state our results we first recall definition of the *homogeneous* Besov space $\dot{B}^s_{p,q}$. Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley decomposition. We take a function $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\phi = \{\xi \in \mathbb{R}^n;\ 1/2 \le |\xi| \le 2\}$ such that $\sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1$ for all $\xi \neq 0$. The functions φ_j is defined by

$$\mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi),$$

where $\mathcal{F}$ denotes the Fourier transform. For $s \in \mathbb{R}$ and $1 \leqq p, q \leqq \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} \equiv \{f \in \mathcal{S}';\ \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \begin{cases} \left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|_{L^p})^q\right)^{1/q}, & 1 \leqq q < \infty, \\ \sup_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|_{L^p}), & q = \infty. \end{cases}$$

Let us denote by P the Helmholtz projection from L^p, $1 < p < \infty$ onto the subspace of solenoidal vector fields PL^p as a bounded operator. It is well-known that P is expressed as

$$P = (P_{jk})_{1\leqq j,k\leqq n}, \quad P_{jk} = \delta_{jk} + R_jR_k, \quad j,k = 1,\dots,n, \tag{1.5}$$

where $\{\delta_{jk}\}_{1\leqq j,k\leqq n}$ is the Kronecker symbol and $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$, $j = 1,\cdots,n$ are the Riesz transforms. One of the advantage of homogeneous Besov spaces $\dot{B}^s_{p,q}$ stems from the fact that the Helmholtz projection P is bounded even for $p = 1$ and $p = \infty$. Indeed, we have the following proposition.

Proposition 1.1 (cf. [15]) *Let $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ and $s \in \mathbb{R}$. The Helmholtz projection P defined by (1.5) is bounded from $\dot{B}^s_{p,q}$ onto itself.*

By using the Stokes operator $-P\Delta$ on $P\dot{B}^s_{p,q}$, the original equations (N-S) can be rewritten to the abstract evolution equation:

$$\begin{cases} \dfrac{du}{dt} - \Delta u + P(u\cdot\nabla u) = 0 \quad \text{on } (0,T), \\ u(0) = a, \end{cases} \tag{1.6}$$

where we use the fact that $-P\Delta u = -\Delta P u = -\Delta u$ for u satisfying $\operatorname{div} u = 0$ in the whole space $\mathbb{R}^n$.

Our definition of the mild solution now reads

Definition 1 *Let $a \in \mathcal{S}'$ with $\operatorname{div} a = 0$ in the sense of distribution. A measurable function u on $\mathbb{R}^n \times (0,T)$ for $0 < T \leqq \infty$ is called a mild solution of* (N-S) *on $(0,T)$ if*

(1) *$u \in L^{\alpha,q}(0,T;PL^r)$ for some $n < r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$ and for some $1 \leqq q \leqq \infty$;*

(2) *u satisfies*

$$u(t) = e^{t\Delta}a - \int_0^t P\nabla \cdot e^{(t-\tau)\Delta}(u \otimes u)(\tau)\, d\tau, \quad 0 < t < T. \tag{1.7}$$

It should be noted by the similar estimate to Lemma 2.2 below that the Duhamel term on the R.H.S. in (1.7) belongs to $L^{\alpha,q}(0,T;PL^r)$ provided u satisfies the condition (1) of Definition 1.

Concerning uniqueness of mild solutions, we have the following proposition.

Proposition 1.2 *Let $a \in \mathcal{S}'$ with $\operatorname{div} a = 0$ in the sense of distribution. The mild solution u of* (N-S) *on $(0,T)$ in the class $L^{\alpha,q}(0,T;L^r)$ for some $n < r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$ and for some $1 \leqq q < \infty$ is unique.*

This uniqueness assertion is an immediate consequence of Lemma 2.2. It should be noted that, in the case $q = \infty$, uniqueness holds provided u is small in $L^{\alpha,\infty}(0,\infty;L^r)$.

We first state well-posedness of global solutions to (N-S) for small initial data a.

Theorem 1 *Let $n < p < \infty$.*

(1) *In case $1 \leqq q < \infty$.*

There exists a constant $\delta = \delta(n,p,q) > 0$ such that if $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ satisfies

$$\|a\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} \leqq \delta, \tag{1.8}$$

then there exists a unique mild solution u of (N-S) *on $(0,\infty)$ with the following properties:*

$$u \in BC([0,\infty); \dot{B}^{-1+\frac{n}{p}}_{p,q}), \tag{1.9}$$

$$u \in L^{\alpha,q}(0,\infty; \dot{B}^0_{r,1}) \quad \text{for all } p \leqq r \leqq \infty \text{ and } 2 \leqq \alpha < \infty \text{ satisfying } \frac{2}{\alpha} + \frac{n}{r} = 1, \tag{1.10}$$

$$t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{p})}u(\cdot) \in BC([0,\infty); \dot{B}^0_{p,1}), \tag{1.11}$$

$$\lim_{t \to +0} \|u(t) - a\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} = 0, \tag{1.12}$$

$$\lim_{t \to +0} t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{p})}\|u(t)\|_{\dot{B}^0_{p,1}} = 0, \tag{1.13}$$

$$\lim_{t \to \infty} \|u(t)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} = 0. \tag{1.14}$$

(2) *In case $q = \infty$.*

Under the condition (1.8)*, there exists a mild solution of* (N-S) *on $(0,\infty)$ with the following properties:*

$$u \in BC_w([0,\infty); \dot{B}^{-1+\frac{n}{p}}_{p,\infty}), \tag{1.15}$$

$$u \in L^{\alpha,\infty}(0,\infty; \dot{B}^0_{r,1}) \quad \text{for all } p \leqq r \leqq \infty \text{ and } 2 \leqq \alpha < \infty \text{ satisfying } \frac{2}{\alpha} + \frac{n}{r} = 1, \tag{1.16}$$

$$t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{p})}u(\cdot) \in BC((0,\infty); \dot{B}^0_{p,1}), \tag{1.17}$$

$$(u(t),\varphi) \to (a,\varphi) \quad \text{for all } \varphi \in \dot{B}^{1-\frac{n}{p}}_{p',1} \quad \text{as } \ t \to +0. \tag{1.18}$$

Here BC_w denotes the class of bounded weakly-star continuous functions.

Concerning the uniqueness, for every $n < r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$, there is a constant $\eta = \eta(n, r)$ such that if u and u' are mild solutions of (N-S) *on* $(0, \infty)$ *in the class* $L^{\alpha,\infty}(0, \infty; PL^r)$ *satisfying*

$$\|u\|_{L^{\alpha,\infty}(0,\infty;\,L^r)} \leqq \eta, \quad \|u'\|_{L^{\alpha,\infty}(0,\infty;\,L^r)} \leqq \eta, \tag{1.19}$$

then it holds that $u \equiv u'$.

Remark 1 (1) Since $\dot{B}^0_{r,1} \subset L^r$, our class (1.10) shows that the solution u given by Theorem 1 belongs to the Serrin class $L^{\alpha,q}(0, \infty; L^r)$, and so by Proposition 1.2, uniqueness holds provided $1 \leqq q < \infty$.
(2) The decay property (1.14) of u in the same space $\dot{B}^{-1+\frac{n}{p}}_{p,q}$ as the initial data a is the corresponding result to what is stated at the end of Kato [11, Note] in the sense that the solution of (N-S) behaves like $\lim_{t\to\infty} \|u(t)\|_{L^n} = 0$ for the initial data $a \in L^n$. On the other hand, in case $q = \infty$ as in Theorem 1 (2), we do not have any corresponding decay to (1.14) since C_0^∞ is not dense in $\dot{B}^{-1+\frac{n}{p}}_{p,\infty}$.

The next theorem shows that the class of initial data is necessarily characterized in scaling invariant homogeneous Besov space when the mild solution u belongs to the Serrin class.

Theorem 2 *Let $a \in \mathcal{S}'$ and* $\operatorname{div} a = 0$ *in the sense of distribution. Suppose that u is a mild solution of* (N-S) *on* $(0, \infty)$ *in* $L^{\alpha,q}(0, \infty; L^r)$ *for some $n < r \leqq \infty$ and some $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$, and for some $1 < q \leqq \infty$. Then it holds necessarily that $a \in P\dot{B}^{-1+\frac{n}{r}}_{r,q}$.*

We obtained the result on analyticity of mild solutions.

Theorem 3 *Let $n < p < \infty$ and $1 \leqq q \leqq \infty$. Suppose that $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ satisfies* (1.8). *The mild solution u of* (N-S) *on* $(0, \infty)$ *given by Theorem 1 is smooth in the space variable as $D^\alpha u(\cdot, t) \in L^\infty$, $0 < t < \infty$ for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ with the estimate*

$$\sup_{0<t<\infty} t^{\frac{1}{2}+\frac{|\alpha|}{2}} \|D^\alpha u(t)\|_{L^\infty} \leqq CK^{|\alpha|}|\alpha|^{|\alpha|}, \tag{1.20}$$

with an absolute constant K, where $C = C(n, p, q)$. In particular, such a mild solution $u(x, t)$ is uniformly analytic in $x \in \mathbb{R}^n$, namely

$$u(x, t) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0, t)}{k!}(x - x_0)^\alpha, \quad 0 < t < \infty \tag{1.21}$$

for all $x_0, x \in \mathbb{R}^n$ with $|x - x_0| < \frac{\sqrt{t}}{eK}$.

The next theorem shows local well-posedness of mild solutions to (N-S) for arbitrary large initial data and its analyticity.

Theorem 4 *Let $n < p < \infty$.*
(1) *In case $1 \leqq q < \infty$.*

For every $a \in P\dot{B}_{p,q}^{-1+\frac{n}{p}}$ there exist $T = T(n,p,q) > 0$ and a unique mild solution u of (N-S) *on $(0,T)$ with the properties:*

$$u \in C([0,T); \dot{B}_{p,q}^{-1+\frac{n}{p}}), \tag{1.22}$$

$$u \in L^{\alpha,q}(0,T;\dot{B}^0_{r,1}) \quad \text{for all } p \leqq r \leqq \infty \text{ and } 2 \leqq \alpha < \infty \text{ satisfying } \tfrac{2}{\alpha} + \tfrac{n}{r} = 1, \tag{1.23}$$

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} u(\cdot) \in BC([0,T); \dot{B}^0_{p,1}), \qquad \lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} \|u(t)\|_{\dot{B}^0_{p,1}} = 0, \tag{1.24}$$

$$\lim_{t\to+0} \|u(t) - a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = 0. \tag{1.25}$$

Such a mild solution u satisfies $D^\alpha u(\cdot,t) \in L^\infty$, $0 < t \leqq T$ for all $\alpha \in \mathbb{N}_0^n$ with the estimate

$$\sup_{0<t<T} t^{\frac{1}{2}+\frac{|\alpha|}{2}} \|D^\alpha u(t)\|_\infty \leqq C K^{|\alpha|} |\alpha|^{|\alpha|} \tag{1.26}$$

with an absolute constant K, where $C = C(n,p,q)$ is independent of T. In particular, $u(x,t)$ is uniformly analytic in $x \in \mathbb{R}^n$.
(2) *In case $q = \infty$.*

There is a constant $\delta' = \delta'(n,p) > 0$ such that if $a \in P\dot{B}_{p,\infty}^{-1+\frac{n}{p}}$ satisfies

$$\sup_{N\leqq j} 2^{(-1+\frac{n}{p})j} \|\varphi_j * a\|_{L^p} \leqq \delta' \tag{1.27}$$

for some $N \in \mathbb{Z}$, then there exist $T = T(a,p) > 0$ and a mild solution u of (N-S) *on $(0,T)$ with the properties:*

$$u \in BC_w([0,T); \dot{B}_{p,\infty}^{-1+\frac{n}{p}}), \tag{1.28}$$

$$u \in L^{\alpha,\infty}(0,T;\dot{B}^0_{r,1}) \quad \text{for all } p \leqq r \leqq \infty \text{ and } 2 \leqq \alpha < \infty \text{ satisfying } \tfrac{2}{\alpha} + \tfrac{n}{r} = 1, \tag{1.29}$$

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} u(\cdot) \in BC((0,T); \dot{B}^0_{p,1}), \tag{1.30}$$

$$(u(t),\varphi) \to (a,\varphi) \quad \text{for all } \varphi \in \dot{B}_{p',1}^{1-\frac{n}{p}} \quad \text{as } t \to +0. \tag{1.31}$$

The uniqueness of mild solution in the class (1.22) *holds under the hypothesis* (1.19) *as in Theorem 1. The analyticity of u remains true in the same way as in* (1.26).

The final result shows that if the solution belongs to the Serrin class globally, then the solution is analytic in $\mathbb{R}^n$ and belongs to more better class.

Corollary 1 *Let $a \in \mathcal{S}'$ and $\operatorname{div} a = 0$ in the sense of distribution. Suppose that u is a mild solution of* (N-S) *on $(0,\infty)$ in $L^{\alpha,q}(0,\infty;L^r)$ for some $n < r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$ and $1 < q < \infty$. Then it holds that $a \in P\dot{B}_{r,q}^{-1+\frac{n}{r}}$ and*

$$u \in L^{\alpha,q}(0,\infty;L^r) \cap L^{\theta,q}(0,\infty;\dot{B}^0_{\beta,1}) \tag{1.32}$$

for all $r \leqq \beta < \infty$ and $2 \leqq \theta \leqq \alpha$ satisfying $\frac{2}{\theta} + \frac{n}{\beta} = 1$. Moreover, $u(t)$ is analytic in $\mathbb{R}^n$ for $t > 0$ as in (1.21).

2 Outline of the Proof

In this article, we state outline of the proof of the results in Section 1. The full proof is stated in the paper [14].

2.1 Key lemmata

The following lemma plays a key role for the proof of Theorem 2 and Corollary 1.

Lemma 2.1 (1) *Let $n < p < \infty$ and $1 \leqq q \leqq \infty$. For $a \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ it holds that $e^{t\Delta}a \in L^{\alpha,q}(0,\infty;\dot{B}^0_{r,1})$ for all $p \leqq r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha}+\frac{n}{r}=1$ with the estimate*

$$\Big\| \|e^{t\Delta}a\|_{\dot{B}^0_{r,1}} \Big\|_{L^{\alpha,q}(0,\infty)} \leqq C\|a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}, \tag{2.1}$$

where $C = C(n,p,q,r)$. In particular, if $a \in \dot{B}_{p,s}^{-1+\frac{n}{p}}$ for $\frac{2}{s}+\frac{n}{p}=1$ with $n < p < \infty$, then it holds that $e^{t\Delta}a \in L^s(0,\infty;\dot{B}^0_{p,1})$.

(2) *Assume that $a \in \mathcal{S}'$ satisfies*

$$e^{t\Delta}a \in L^{\alpha,q}(0,\infty;L^r).$$

for $n < r \leqq \infty$ and $2 \leqq \alpha < \infty$ with $\frac{2}{\alpha}+\frac{n}{r}=1$ and for $1 < q \leqq \infty$. Then it holds that $a \in \dot{B}_{r,q}^{-1+\frac{n}{r}}$ with the estimate

$$\|a\|_{\dot{B}_{r,q}^{-1+\frac{n}{r}}} \leqq C\|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)}, \tag{2.2}$$

where $C = C(n,r,q)$.

In case of $\alpha = q$, the result is in [2, Theorem 2.34].

We define the nonlinear term

$$N(u,v) = \int_0^t e^{(t-\tau)\Delta}P(u\cdot\nabla v)(\tau)\,d\tau = \int_0^t P\nabla\cdot e^{(t-\tau)\Delta}(u\otimes v)(\tau)\,d\tau$$

for solenoidal vector fields u and v. The next lemma shows bilinear estimates which will be used to control the nonlinear term $N(u,v)$.

Lemma 2.2 (1) *Let $n < r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfy $\frac{2}{\alpha}+\frac{n}{r}=1$. Let $1 \leqq q \leqq \infty$. Assume that $u \in L^{\alpha,q}(0,T;\dot{B}^0_{r,1})$. Then for every $v \in L^{\theta,\tilde{q}}(0,T;\dot{B}^0_{\beta,1})$ with $r' \leqq \beta \leqq \infty$, $\alpha' \leqq \theta < \infty$ and $1 \leqq \tilde{q} \leqq \infty$, it holds that $N(u,v) \in L^{\theta,\tilde{q}}(0,T;\dot{B}^0_{\beta,1})$ with the estimate*

$$\|N(u,v)\|_{L^{\theta,\tilde{q}}(0,T;\dot{B}^0_{\beta,1})} \leqq C\|u\|_{L^{\alpha,q}(0,T;\dot{B}^0_{r,1})}\|v\|_{L^{\theta,\tilde{q}}(0,T;\dot{B}^0_{\beta,1})} \tag{2.3}$$

for all $0 < T \leqq \infty$, where $C = C(n,r,q,\theta,\beta,\tilde{q})$ is independent of T. In particular, we may take $\beta = r$ and $\theta = \alpha$, and hence for $u, v \in L^{\alpha,q}(0,T;\dot{B}^0_{r,1})$, it holds that

$$\|N(u,v)\|_{L^{\alpha,q}(0,T;\dot{B}^0_{r,1})} \leqq C\|u\|_{L^{\alpha,q}(0,T;\dot{B}^0_{r,1})}\|v\|_{L^{\alpha,q}(0,T;\dot{B}^0_{r,1})} \tag{2.4}$$

(2) *Let* $n < p < \infty$ *and* $1 \leqq q \leqq \infty$. *We assume that* $\sup_{0<t<\infty} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_{\dot{B}^0_{p,1}} < \infty$ *and* $\sup_{0<t<\infty} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|v(t)\|_{\dot{B}^0_{p,1}} < \infty$. *Then it holds that*

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|N(u,v)(t)\|_{\dot{B}^0_{p,1}} \leqq C(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(\tau)\|_{\dot{B}^0_{p,1}})(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|v(\tau)\|_{\dot{B}^0_{p,1}}), \tag{2.5}$$

$$\|N(u,v)(t)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} \leqq C(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(\tau)\|_{\dot{B}^0_{p,1}})(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|v(\tau)\|_{\dot{B}^0_{p,1}}) \tag{2.6}$$

for all $0 < t \leqq \infty$, *where* $C = C(n,p)$ *in* (2.5) *and* $C = C(n,p,q)$ *in* (2.6) *are independent of* t.

2.2 Proof of Theorem 1 (1); in case $1 \leqq q < \infty$

For the proof of Theorem 1 (1), we make use of the implicit function theorem for Banach spaces. Let $n < p < \infty$ and $2 < s < \infty$ satisfy $\frac{2}{s} + \frac{n}{p} = 1$. Let $1 \leqq q < \infty$. We define the class X of mild solutions by

$$\begin{aligned} X = \Big\{ & u \in BC([0,\infty); \dot{B}^{-1+\frac{n}{p}}_{p,q}) \cap L^{s,q}(0,\infty; \dot{B}^0_{p,1}) \cap L^{2,q}(0,\infty; \dot{B}^0_{\infty,1}); \\ & t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}u(\cdot) \in BC([0,\infty); \dot{B}^0_{p,1}); \ \lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_{\dot{B}^0_{p,1}} = 0 \Big\} \end{aligned} \tag{2.7}$$

with the norm

$$\|u\|_X \equiv \sup_{0<t<\infty} \|u(t)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} + \|u\|_{L^{s,q}(0,\infty;\dot{B}^0_{p,1})} + \|u\|_{L^{2,q}(0,\infty;\dot{B}^0_{\infty,1})} + \sup_{0<t<\infty} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_{\dot{B}^0_{p,1}}.$$

We define a map $G(a,u)$ by

$$G(a,u)(t) = u(t) - e^{t\Delta}a + N(u,u)(t) \quad 0 < t < \infty, \tag{2.8}$$

for $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ and $u \in X$. Then we have

Lemma 2.3 *Let* $n < p < \infty$ *and* $2 < s < \infty$ *satisfy* $\frac{2}{s} + \frac{n}{p} = 1$, *and let* $1 \leqq q < \infty$.

(1)

$$G : \ P\dot{B}^{-1+\frac{n}{p}}_{p,q} \times X \ni (a,u) \mapsto G(a,u) \in X$$

is a continuous mapping.

(2) *For each* $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,q}$, *the map* $G(a,\cdot)$ *is of class* C^1 *from* X *into itself and the Fréchet derivative* $G_u(a,u) \in \mathcal{B}(X)$ *is given by*

$$\begin{aligned} G_u(a,u)v(t) &= v(t) + \int_0^t e^{(t-\tau)\Delta}P(u\cdot\nabla v + v\cdot\nabla u)(\tau)\,d\tau \\ &= v(t) + N(u,v)(t) + N(v,u)(t), \quad 0 < t < \infty \end{aligned}$$

for $v \in X$. *Here* $\mathcal{B}(X)$ *denotes the set of bounded linear operators on* X.

Now we prove Theorem 1 (1). Let $n < p \leqq r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfy $\frac{2}{\alpha} + \frac{n}{r} = 1$. Let $1 \leqq q < \infty$. Assume that $a \in P\dot{B}_{p,q}^{-1+\frac{n}{p}}$.

Step 1. First we prove existence of the solution in the class (1.9). Our aim is to solve the equation $G(a, u) = 0$ by representing $u \in X$ in terms of $a \in P\dot{B}_{p,q}^{-1+\frac{n}{p}}$. Since

$$G(0, 0) = 0, \quad G_u(0, 0) = I_X \quad (\textit{the identity map on } X),$$

it follows from the implicit function theorem on Banach spaces that there exist positive constants δ, ϵ and a continuous map $u : U_\delta \to V_\epsilon$ with

$$U_\delta = \{a \in P\dot{B}_{p,q}^{-1+\frac{n}{p}};\ \|a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leqq \delta\}, \quad V_\varepsilon = \{u \in X;\ \|u\|_X \leqq \varepsilon\}.$$

such that the equation $G(a, u) = 0$ can be uniquely expressed as $u = u[a]$ for $a \in U_\delta$. Obviously such a constant δ coincides with that of (1.8), and the continuous map u is the desired mild solution of (N-S) on $(0, \infty)$ for the initial data a satisfying (1.8). Apparently, we see that $u = u[a]$ belongs to X. By interpolation, it holds that

$$\|u\|_{L^{\alpha,q}(0,\infty;\dot{B}^0_{r,1})} \leqq C\|u\|^{\frac{p}{r}}_{L^{s,q}(0,\infty;\dot{B}^0_{p,1})}\|u\|^{1-\frac{p}{r}}_{L^{2,q}(0,\infty;\dot{B}^0_{\infty,1})}$$

for all $p \leqq r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$ with $C = C(n, p, q, r)$, from which we obtain (1.10). The fact that u satisfies (1.9), (1.11) and (1.13) is due to definition (2.7) of the space X. Since $u[a](t) = e^{t\Delta}a - N(u, u)(t)$, we see that the property (1.12) is a consequence of

$$\lim_{t\to+0} \|e^{t\Delta}a - a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = 0, \quad \lim_{t\to+0} \|N(u, u)(t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = 0.$$

Step 2. For the proof of (1.14), we consider the following auxiliary evolution equation.

$$\begin{cases} \dfrac{dv}{dt} - \Delta v + P(u \cdot \nabla v) = 0 \quad \text{on } t > 0, \\ v(0) = b. \end{cases} \tag{2.9}$$

Concerning the unique existence of mild solution v for (2.9), we have the following lemma.

Lemma 2.4 *Let $n < p \leqq r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfy $\frac{2}{\alpha} + \frac{n}{r} = 1$, and let $\frac{np}{2p-n} < \beta < n$. There exists a constant $\eta = \eta(n, p, r, q, \beta)$ such that if $u \in X$ satisfies*

$$\|u\|_X \leqq \eta \tag{2.10}$$

then for every $b \in P\dot{B}_{p,q}^{-1+\frac{n}{p}} \cap L^\beta$, there exists a unique mild solution v of (2.9) in the class $X \cap BC([0, \infty); L^\beta)$, i.e.,

$$v(t) = e^{t\Delta}b - N(u, v)(t), \quad 0 < t < \infty. \tag{2.11}$$

Moreover, such a mild solution v satisfies

$$\lim_{t\to\infty} \|v(t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = 0. \tag{2.12}$$

An immediate consequence of Lemma 2.4 with the aid of uniqueness of solution to (2.9) is a decay of the mild solution of (N-S) with $a \in \dot{B}_{p,q}^{-1+\frac{n}{p}} \cap L^\beta$. More precisely, we have the following lemma.

Lemma 2.5 *Let $n < p \leqq r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfy $\frac{2}{\alpha} + \frac{n}{r} = 1$, and let $1 \leqq q < \infty$. Assume that $\frac{np}{2p-n} < \beta < n$. There exists a constant $\delta' = \delta'(n, p, r, q, \beta)$ such that if $a \in P\dot{B}_{p,q}^{-1+\frac{n}{p}} \cap L^\beta$ satisfies $\|a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leqq \delta'$, then the mild solution u given by Step 1 has the additional property that $u \in X \cap BC([0,\infty); L^\beta)$ with*

$$\lim_{t\to\infty} \|u(t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = 0. \tag{2.13}$$

Now, we are in position to prove (1.14). We take β so that

$$\frac{np}{2p-n} < \beta < n$$

and fix such a β. Let $\delta' = \delta'(n, p, r, q, \beta)$ be as in Lemma 2.5. Define $\lambda = \lambda(n, p, r, q, \beta) \equiv \frac{\delta'}{2}$. In such a situation, assuming $a \in P\dot{B}_{p,q}^{-1+\frac{n}{p}}$ with $\|a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leqq \lambda$, we may prove that

$$\lim_{t\to\infty} \|u[a](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = 0, \tag{2.14}$$

where $U_\delta \ni a \mapsto u[a] \in V_\varepsilon$ is the solution map defined in Step 1. By continuity of the map $u[\cdot]$, for every $a \in P\dot{B}_{p,q}^{-1+\frac{n}{p}}$ with $\|a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leqq \lambda$ and for every $\varepsilon > 0$, there is a constant $\kappa = \kappa(\varepsilon, a, p, n, q, r)$ such that if $a_\varepsilon \in P\dot{B}_{p,q}^{-1+\frac{n}{p}}$ satisfies $\|a - a_\varepsilon\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leqq \kappa$, then it holds that $\|u[a] - u[a_\varepsilon]\|_X \leqq \varepsilon$. Since $P\dot{B}_{p,q}^{-1+\frac{n}{p}} \cap L^\beta$ is dense in $P\dot{B}_{p,q}^{-1+\frac{n}{p}}$, we may assume that $a_\varepsilon \in P\dot{B}_{p,q}^{-1+\frac{n}{p}} \cap L^\beta$ with $\|a_\varepsilon\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leqq \delta'$. Hence it follows from Lemma 2.5 that

$$\lim_{t\to\infty} \|u[a_\varepsilon](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = 0. \tag{2.15}$$

Since

$$\begin{aligned}
\|u[a](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} &\leqq \|u[a](t) - u[a_\varepsilon](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} + \|u[a_\varepsilon](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \\
&\leqq \|u[a] - u[a_\varepsilon]\|_X + \|u[a_\varepsilon](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \\
&\leqq \varepsilon + \|u[a_\varepsilon](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}
\end{aligned}$$

holds for all $t \in (0,\infty)$, we obtain from (2.15) that

$$\overline{\lim_{t\to\infty}} \|u[a](t)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude (2.14). This completes the proof of Theorem 1 (1). The proof of Theorem 1 (2) in case $q = \infty$ is omitted (cf. [14]).

2.3 Proof of Theorem 2

Let $n < r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfy $\frac{2}{\alpha} + \frac{n}{r} = 1$. Let $1 < q \leqq \infty$. We assume that $u(t) = e^{t\Delta}a - N(u,u)(t)$ is the mild solution of (N-S) on $(0,\infty)$ in $L^{\alpha,q}(0,\infty; L^r)$. A slight modification of Lemma 2.2 (1) from $\dot{B}^0_{r,1}$ to L^r yields that $N(u,u) \in L^{\alpha,q}(0,\infty; L^r)$. Therefore it holds that $e^{t\Delta}a \in L^{\alpha.q}(0,\infty; L^r)$, and hence from Lemma 2.1 (2), we conclude that $a \in P\dot{B}^{-1+\frac{n}{r}}_{r,q}$. This proves Theorem 2.

2.4 Proof of Theorem 3

The proof of Theorem 3 is rather long and complicated, so we omit it. For details, see [14].

2.5 Proof of Theorem 4

For construction of the mild solution locally on some interval $(0,T)$ for an arbitrary initial data $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,q}$, we make use of the successive approximation $\{u_j\}_{j=0}^{\infty}$ as

$$\begin{aligned} u_{j+1}(t) &= u_0(t) - N(u_j,u_j)(t), \quad j = 0,1,2\ldots, \\ u_0(t) &= e^{t\Delta}a, \quad N(u_j,u_j)(t) = \int_0^t e^{(t-\tau)\Delta} P\nabla\cdot(u_j \otimes u_j)(\tau)\,d\tau. \end{aligned}$$

Let us define $M_j = M_j(t)$ by

$$M_j(T) = \sup_{0<t<T} \|u_j(t)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} + \|u_j\|_{L^{s,q}(0,T;\dot{B}^0_{p,1})} + \|u_j\|_{L^{2,q}(0,T;\dot{B}^0_{\infty,1})} + \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u_j(t)\|_{\dot{B}^0_{p,1}},$$

for $j = 0,1,2\ldots$, where s is chosen as $\frac{2}{s} + \frac{n}{p} = 1$. Since $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,q}$, by the heat semigroup estimates in the homogeneous Besov spaces (cf. [13, Lemma 2.2]) and (2.1) it holds that $M_0 < \infty$. Assume that $M_j < \infty$. Then by Lemma 2.2 we have

$$M_{j+1} \leqq M_0 + CM_j^2, \tag{2.16}$$

where $C = C(n,p,q)$ is independent of j. Hence by induction we have $M_j < \infty$ for all $j = 0,1,2\ldots$. Now, we take M_0 in such a way that

$$M_0 = M_0(T) < \frac{1}{4C}. \tag{2.17}$$

This is fulfilled by choosing T small enough since $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ for $1 \leqq q < \infty$. In case $q = \infty$ under the hypothesis (1.27) on $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,\infty}$, we may also take a small $T > 0$ so that (2.17) is fulfilled. For more details, see [15, Theorem 1.3 (2)]. Then under the condition (2.17), we obtain from (2.16) that

$$M_j \leqq \frac{1-\sqrt{1-4CM_0}}{2C} := M \quad \text{for } j = 0,1,2\ldots.$$

By the standard method as in Fujita-Kato [8] and Kato [11], we see that such a bound M of $\{M_j\}_{j=0}^{\infty}$ yields the mild solution u of (N-S) in the class

$$u \in \begin{cases} BC([0,T); \dot{B}_{p,q}^{-1+\frac{n}{p}}) \cap L^{s,q}(0,T; \dot{B}_{p,1}^{0}) \cap L^{2,q}(0,T; \dot{B}_{\infty,1}^{0}), & 1 \leqq q < \infty, \\ BC_w([0,T); \dot{B}_{p,\infty}^{-1+\frac{n}{p}}) \cap L^{s,\infty}(0,T; \dot{B}_{p,1}^{0}) \cap L^{2,\infty}(0,T; \dot{B}_{\infty,1}^{0}), & q = \infty, \end{cases}$$

provided the condition (2.17) is fulfilled. The properties (1.22)–(1.25) for $1 \leqq q < \infty$ and those (1.28)–(1.31) for $q = \infty$ are proved as similar manner to the proof of (1.9)–(1.13) and (1.15)–(1.18) in Theorem 1, respectively. Analyticity is proved in the same way as the proof of Theorem 3, where the time interval is restricted to $0 < t < T$.

2.6 Proof of Corollary 1

By Theorem 2, it follows that $a \in P\dot{B}_{r,q}^{-1+\frac{n}{r}}$. Applying Theorem 4, there exists $T > 0$ and a unique mild solution $\tilde{u}(t)$ of (N-S) on $(0,T)$ in the class $C([0,T); \dot{B}_{r,q}^{-1+\frac{n}{r}}) \cap L^{\theta,q}(0,T; \dot{B}_{\beta,1}^{0})$ with $t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}u(t) \in BC([0,T); \dot{B}_{r,1}^{0})$ for all $r \leqq \beta \leqq \infty$ and $2 \leqq \theta \leqq \alpha$ satisfying $\frac{2}{\theta} + \frac{n}{\beta} = 1$. Then it follows that uniqueness of mild solutions in the class $L^{\alpha,q}(0,T : L^r)$ that $u(t) = \tilde{u}(t)$ for $0 \leqq t < T$.

We next show that there is some $T < T_0 < \infty$ such that $u \in L^{\theta,q}(T_0, \infty; \dot{B}_{\beta,1}^{0})$ for all β and θ as above. Indeed, since $u \in L^{\alpha,q}(0,\infty; L^r)$, for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$\|u\|_{L^{\alpha,q}(T_\varepsilon,\infty;L^r)} \leqq \varepsilon. \tag{2.18}$$

Since u is a mild solution of (N-S) on $(0,\infty)$, we have an expression as

$$u(t) = e^{(t-T_\varepsilon)\Delta}u(T_\varepsilon) - \int_{T_\varepsilon}^{t} P\nabla \cdot e^{(t-\tau)\Delta}(u \otimes u)(\tau)\, d\tau.$$

Hence by (2.18) and a similar argument to Lemma 2.2 (1) it holds that

$$\begin{aligned} \|e^{(t-T_\varepsilon)\Delta}u(T_\varepsilon)\|_{L^{\alpha,q}(T_\varepsilon,\infty;L^r)} &\leqq \|u\|_{L^{\alpha,q}(T_\varepsilon,,\infty;L^r)} + C\|u\|^2_{L^{\alpha,q}(T_\varepsilon,,\infty;L^r)} \\ &\leqq C\varepsilon, \end{aligned}$$

where $C = C(n,r,q)$ is independent of T_ε. Then it follow from Lemma 2.1(2) that

$$\|u(T_\varepsilon)\|_{\dot{B}_{r,q}^{-1+\frac{n}{r}}} \leqq C\|e^{(t-T_\varepsilon)\Delta}u(T_\varepsilon)\|_{L^{\alpha,q}(T_\varepsilon,\infty;L^r)} \leqq C\varepsilon$$

with he same constant C as above. Taking $\varepsilon > 0$ so small that $C\varepsilon \leqq \delta$ with the same δ as in Theorem 1 with p replaced by r, we obtain a unique mild solution v of (N-S) on (T_ε, ∞) in the class $BC([T_\varepsilon,\infty); \dot{B}_{r,q}^{-1+\frac{n}{r}}) \cap L^{\theta,q}(T_\varepsilon,\infty; \dot{B}_{\beta,1}^{0})$ with $v(T_\varepsilon) = u(T_\varepsilon)$ for all $r \leqq \beta \leqq \infty$ and $2 \leqq \theta \leqq \alpha$ satisfying $\frac{2}{\theta} + \frac{n}{\beta} = 1$. Hence again by uniqueness, it holds that

$$u(t) = v(t) \quad \text{for} \quad T_\varepsilon \leqq t < \infty,$$

which yields that

$$u \in L^{\theta,q}(T_\varepsilon, \infty; \dot{B}_{\beta.1}^{0}) \tag{2.19}$$

for all β and θ as above.

Now, our remaining task is to show that the local existence times T is able to be prolonged up to T_ε in (2.19). We prove it by contradiction. Assume that there is $0 < T_* < T_\varepsilon$ such that u cannot be extended beyond T_*. Then it follows from Giga [9, Theorem 4] that there exists $\eta > 0$ such that

$$\|u(t)\|_{L^r} \geq C(T_* - t)^{-\frac{r-n}{2r}}, \quad T_* - \eta < t < T_*$$

with some positive constant $C = C(n, r)$. It is easy to verify that

$$\begin{aligned}(T_* - t)^{-\frac{r-n}{2r}} &\in L^{\alpha,\infty}(0, T_*)\\ &\notin L^{\alpha,q}(0, T_*)\end{aligned}$$

for $1 < q < \infty$, which implies that $\|u(t)\|_{L^r} \notin L^{\alpha,q}(0, T_*)$. This causes a contradiction and hence we obtain that

$$u \in L^{\theta,q}(0, T; \dot{B}^0_{\beta,1}) \tag{2.20}$$

for all $0 < T < \infty$. Now it follows from (2.19) and (2.20) that

$$u \in L^{\theta,q}(0, \infty; \dot{B}^0_{\beta,1})$$

for all $r \leqq \beta \leqq \infty$ and $2 \leqq \theta \leqq \infty$ satisfying $\frac{2}{\theta} + \frac{n}{\beta} = 1$. Thus the solution $u(t)$ coincides with the mild solution of (N-S) given by Theorem 1 so that it has the properties (1.9)-(1.14). Furthermore, it follows from Theorem 3 that $u(t)$ is analytic in $\mathbb{R}^n$ for all $0 < t < \infty$. This completes the proof of Corollary 1.

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